

On the theorem of Duminil-Copin and Smirnov about the number of self-avoiding walks in the hexagonal lattice

Martin Klazar*

January 13, 2013

Abstract

This is an exposition of the theorem from the title, which says that the number of self-avoiding walks with n steps in the hexagonal lattice has asymptotics $(2 \cos(\pi/8))^{n+o(n)}$. We lift the key identity to formal level and simplify the part of the proof bounding the growth constant from below. In our calculation the lower bound comes from an identity asserting that a linear combination of 288 generating functions counting self-avoiding walks in a certain domain by length, final edge direction and winding number modulo 48 equals the geometric series $\sum_{n \geq 1} (2 \cos(\pi/8))^n x^n$.

1 Introduction

Let s_n be the number of n -step self-avoiding walks with fixed initial vertex in the hexagonal lattice graph, which is a plane graph whose vertices and edges are the vertices and unit segments in the tiling of the plane into regular hexagons. In particular, every vertex has degree three. An argument using submultiplicativity of s_n proves existence of the limit $\mu = \lim s_n^{1/n}$; clearly, $1 \leq \mu \leq 2$. In 1982, Nienhuis [5] conjectured that $\mu = \sqrt{2 + \sqrt{2}}$. His conjecture was recently proved by Duminil-Copin and Smirnov [1]. (For more information on counting self-avoiding walks in lattices see [2].)

Theorem 1 (Duminil-Copin and Smirnov, 2010) *The number s_n of self-avoiding walks in the hexagonal lattice with length n and fixed initial vertex has for $n \rightarrow \infty$ asymptotics*

$$s_n = \mu^{n+o(n)}, \quad \mu = 2 \cos(\pi/8) = \sqrt{2 + \sqrt{2}} = 1.84775 \dots$$

*Department of Applied Mathematics, Charles University, Faculty of Mathematics and Physics, Malostranské náměstí 25, 118 00 Praha, Czech Republic, klazar@kam.mff.cuni.cz

Here we give an exposition of this remarkable result, the proof of which uses a beautiful combination of geometric and generating functions based ideas.

My motivation as an enumerative combinatorialist fond of generating functions was to understand the arguments of [1] from this angle. Pondering the identities in Lemmas 1 and 2 of [1] one sees that they can be obtained as specializations of a more general formal identity. We present this generalization in Proposition 2. The formal approach treats finite and infinite domains of vertices on equal footing and skips some limit transitions of [1]. The lower bound on μ comes from it in a more direct way compared to [1], by applying the identity of Proposition 2 to an appropriate domain. For this domain we get a formal identity stated in Corollary 6: If one pigeonholes the length generating function of self-avoiding walks in the domain (with fixed starting edge) according to the six possible directions of the final edge and the winding number modulo 48, then a linear combination of these 288 generating functions equals the geometric series $\sum_{n \geq 1} (2 \cos(\pi/8))^n x^n$. In deriving the upper bound on μ in Section 3 we follow the arguments of [1] more closely. Finally, in Section 4 we prove results on winding number required for the proof, such as that the winding number of closed self-avoiding walks attains only two values.

We adapt the notation of [1] to our needs. For example, the hexagonal graph H we work with is not exactly the hexagonal lattice graph but its isomorphic embedding in the square lattice graph. We found it more convenient to think in terms of rectangular pictures and, for exposing the underlying combinatorial reasoning, to divorce the hexagonal lattice graph from its particular embedding in the complex plane. This is the reason for using the direction labelling ℓ that just produces the complex vectors $v - u$ for oriented edges uv of the hexagonal lattice graph in \mathbb{C} . Also, instead of the self-avoiding walks between mid-edges of [1] we work with self-avoiding walks with ordinary edges and thus allow coincidence of the final vertex with an earlier vertex of the walk.

2 A formal identity and the lower bound on μ

We begin with some definitions. Let $G = (V, E)$ be a (simple, undirected) graph, typically infinite but with finite degrees. A *walk* in G is any sequence of vertices and edges $w = (v_1, e_1, v_2, e_2, \dots, e_n, v_{n+1})$ of G such that $n \geq 1$, $e_j = v_j v_{j+1}$ and $e_j \neq e_{j+1}$ (we do not allow backtracks); $n = |w|$ is the *length* of w , v_1 is its *first (initial)* and v_{n+1} *last (final) vertex* and similarly for the extremal edges. The *reversal* w^r of w is $(v_{n+1}, e_n, v_n, e_{n-1}, \dots, e_1, v_1)$. $E(w) = \{e_1, \dots, e_n\}$ is the set of its edges and the final edge is denoted $f(w) = e_n$. If $v_1 = v_{n+1}$ then w is a *closed walk*. If all vertices v_j are distinct (thus w is a path) we say that w is a *self-avoiding walk* or, for short, a *SAW*. A *closed SAW* is a closed walk in which $v_1 = v_{n+1}$ is the only equality of vertices. A *weak SAW* is a walk in G such that all vertices v_j are distinct, with the possible exception of the equality $v_{n+1} = v_j$ for some $j \leq n$. Thus a weak SAW is a SAW or a closed SAW or a concatenation $w_1 w_2$ of a SAW w_1 and a closed SAW w_2 such that $w_1 \cap w_2$ is exactly the final vertex of w_1 , which coincides with the initial/final vertex of

w_2 .

We work with the infinite plane *hexagonal graph*

$$H = (V, E) \subset \mathbb{R}^2, \quad V = \mathbb{Z} \times \mathbb{Z},$$

arising from the square lattice graph by omitting half of the horizontal edges, those whose left endvertex has coordinates with different parity. So H is a rectangular drawing of the hexagonal lattice graph H' ; it is a cubic graph whose all faces are rectangular hexagons. We obtain H from H' by keeping the unit length of edges and the horizontal edges but compressing H' horizontally so that all non-horizontal edges become vertical. For any vertex v we denote by $N(v) = \{e_1, e_2, e_3\}$ the triple of edges incident with v with some fixed counter-clockwise indexing (each $N(v)$ has three of them). By s_n we denote the number of SAWs in H with length n and first vertex equal to $(0, 0)$. As we mentioned, $\mu = \lim s_n^{1/n}$ exists. We present the proof of [1] that $\mu = 2 \cos(\pi/8)$.

Let $X \subset V$ be a finite or infinite connected set of vertices, we call it a *domain*. $H \setminus X$ denotes the graph induced by H on the vertex set $V \setminus X$. We call X a *simple domain* if every connected component of $H \setminus X$ is infinite. $E(X) \subset E$ is the set of edges with an endvertex in X and $\partial X \subset E(X)$ are the *border edges*, the edges with exactly one endvertex in X . A *cycle* C in X is any 2-regular connected subgraph of H with all vertices in X ; $|C|$ is the number of its edges (or vertices). For a domain X and two edges $a \in \partial X$ and $e \in E(X)$ we define $X(a, e)$ to be the set of all weak SAWs $w = (v_1, e_1, \dots, e_n, v_{n+1})$ such that $e_1 = a$, v_1 is the vertex of a not in X , $e_n = e$ and if $v_j \notin X$ then $j = 1$ or $j = n + 1$. Thus $X(a, e)$ is the set of weak SAWs going in X from a to e . If $e \in \partial X$ and the vertices of a and e in $V \setminus X$ are distinct then $X(a, e)$ consists only of SAWs; if they coincide then $X(a, e)$ consists only of closed SAWs. We are interested in the generating function (i.e., formal power series)

$$F_{a,e}(x) = F_{a,e}^X(x) = \sum_{w \in X(a,e)} x^{|w|}.$$

Note that if $a = e$ then $F_{a,a}(x) = x$. For any set of edges $A \subset E(X)$ we define the more general set of weak SAWs $X(a, A) = \bigcup_{e \in A} X(a, e)$ and the corresponding generating function

$$F_{a,A}(x) = \sum_{w \in X(a,A)} x^{|w|} = \sum_{e \in A} F_{a,e}(x).$$

Consider the primitive sixth root of unity

$$\zeta = \exp(2\pi i/6)$$

and two mappings

$$\ell : \overline{E} \rightarrow \{\zeta^j \mid j = 0, 1, 2, 3, 4, 5\} \quad \text{and} \quad r : \text{walks in } H \rightarrow \mathbb{Z}$$

which we define in detail after stating the identity. \overline{E} means oriented edges (marked by bars), every $e \in E$ appears in \overline{E} in its two orientations, and we

extend ℓ to walks by setting $\ell(w) = \ell(\bar{e})$ where $e = f(w)$ and has certain orientation. The mapping r gives winding numbers of walks. Now we state the main identity and afterwards give further explanation.

Proposition 2 *Let $H = (V, E)$ be the hexagonal graph, $X \subset V$ a simple domain and $a \in \partial X$ its border edge. Then we have in $\mathbb{C}[y, y^{-1}][[x]]$ the formal identity*

$$\begin{aligned} \sum_{e \in \partial X} \ell(\bar{e}) y^{r(e)} F_{a,e}(x) &= (1 + x\zeta^{-2}y + x\zeta^2y^{-1}) \sum_{w \in S} \ell(w) y^{r(w)} x^{|w|} + \\ &+ (\zeta^2 y^{-4} + \zeta^{-2} y^4) \sum_{w \in S, C \in C(w)} \ell(w) y^{r(w)} x^{|w|+|C|} \end{aligned}$$

where S is the set of all SAWs that start in a and have all vertices but the first in X and $C(w)$ is the set of all cycles in X that intersect w only in its final vertex.

Here x and y are formal variables and only later we plug in for them some complex numbers.

We define orientation of edges and their direction ℓ . For $e \in E$, an *oriented edge* is a pair $\bar{e} = (e, v)$ where v is an endvertex of e . We think of \bar{e} as oriented away from v . The *basic orientation* is the left-right and bottom-up orientation of edges in E and the *basic direction* ℓ_b is $\ell_b(e) = \zeta^0 = 1$ for every horizontal edge and $\ell_b(e) = \zeta^1$ (resp. $\ell_b(e) = \zeta^2$) if e is vertical and its top vertex is incident with a horizontal edge going to the right (resp. to the left). For general oriented edges we define ℓ by $\ell(\bar{e}) = \ell(e, v) = \ell_b(e)$ if \bar{e} is oriented as in the basic orientation and $\ell(\bar{e}) = \zeta^3 \ell_b(e) = -\ell_b(e)$ if it is oriented in the opposite way. The key features of ℓ are that

$$\ell(e, u) + \ell(e, v) = 0$$

for any edge $e = uv$ and the multiplicative property

$$\ell(\overline{e_{i+1}}) = \zeta^2 \ell(\bar{e}_i), \quad i = 1, 2, 3 \pmod{3},$$

holding for any triple $N(v) = \{e_1, e_2, e_3\}$ if the e_i are oriented in the same way, towards v or away from it. For $e \in \partial X$ we define $\bar{e} = (e, e \cap X)$, so e is oriented away from its vertex in X (this concerns the left side of the identity). Any walk $w = (v_1, e_1, \dots, e_n, v_{n+1})$ has a natural orientation from v_1 to v_{n+1} and we set $\ell(w) = \ell(\bar{e}_n)$ where the final edge is oriented in the opposite way to the natural orientation of w (this concerns summands on the right side of the identity).

We define the winding number r . A *hook* $h = (v_1, e_1, v_2, e_2, v_3) = e_1 e_2$ is any SAW in H with length 2; v_2 is its *central vertex*. We call h a *left hook* if it ‘turns left’, e_2 comes clockwise after e_1 with respect to v_2 ; else, if e_2 comes counter-clockwise after e_1 , we call h a *right hook*. To the left hooks we give weight $r(h) = +1$ and to the right ones $r(h) = -1$. For any walk $w = (v_1, e_1, \dots, e_n, v_{n+1})$ we define its *winding number* by

$$r(w) = \sum_{i=1}^{n-1} r((v_i, e_i, v_{i+1}, e_{i+1}, v_{i+2})).$$

Thus $r(w)$ is the difference between the number of left and right hooks contained in w . When $|w| = 1$ we set $r(w) = 0$. It follows that $\zeta^{r(w)} = \ell(\overline{e_n})/\ell(\overline{e_1})$, with the natural orientation of the walk. We will show that if X is a simple domain and the edges $a, e \in \partial X$ are fixed then all SAWs $w \in X(a, e)$ have the same winding number, which explains the notation $r(e) = r(w)$ on the left side of the identity.

Our explanation of the identity is now complete. Before proving it we review a few properties of winding number used in the proof. Clearly, $r(w^r) = -r(w)$ and if w is split by a vertex v in two walks as $w = w_1 w_2$ then

$$r(w) = r(w_1) + r(w_2) + r(h)$$

where h is the hook in w with central vertex v . If $w = (v_1, e_1, \dots, e_n, v_{n+1} = v_1)$ is a closed walk, it is convenient to define its *complete winding number* r^* as

$$r^*(w) = r(w') = r(w) + r(h)$$

where w' is the extension of w by the edge $e_{n+1} = e_1$ and h is the hook $(v_n, e_n, v_1, e_1, v_2)$. Thus $r^*(w)$ is the sum of weights of the hooks in w when their central vertex runs through *all* vertices of w including the initial/final vertex (the hook of which, strictly speaking, is not contained in w). It is easy to see that for any closed walk $r^*(w) = 6k$ for some $k \in \mathbb{Z}$. An important case is when w is a closed SAW. We recall a particular case of Jordan's curve theorem.

Proposition 3 *Every closed SAW w in H separates the plane in two connected components, the bounded interior of w and the unbounded exterior of w . For any straight segment uv that intersects w in a point different from u, v and the vertices of w , the endpoints u and v lie in distinct components of $\mathbb{R}^2 \setminus w$.*

We say that a closed SAW w is *clockwise* (resp. *counter-clockwise*) oriented if its interior lies to the right (resp. left) of its edges when they are naturally oriented.

Proposition 4 *The following holds.*

1. *If w is a closed SAW in H then, depending on the counter-clockwise or clockwise orientation,*

$$r^*(w) = +6 \quad \text{or} \quad r^*(w) = -6 .$$

2. *If $X \subset V$ is a simple domain and $a, e \in \partial X$ are two boundary edges then all SAWs of $X(a, e)$ have the same winding number.*

We prove Proposition 4 in Section 4. For brevity we do not prove Proposition 3 and simply assume it, although we could prove it by the same inductive argument of Section 4. For more information on Jordan's theorem and its proofs see [3] and [4]. Now we prove the identity.

Proof of Proposition 2. Let H , X and a be as given. Let $X(a) = X(a, E(X))$, the set of all weak SAWs whose first edge is a , first vertex is the endvertex of a not in X and all vertices but the initial one and possibly the final one lie in X . Consider the formal sum

$$\sum_{(w,v) \in X(a) \times X, v \in f(w)} \ell(f(w), v) y^{r(w)} x^{|w|}.$$

For each n there are only finitely many pairs (w, v) in the sum with $|w| = n$. Hence the sum is a well defined element of $\mathbb{C}[y, y^{-1}][[x]]$ since for any ordering of the set of pairs into a sequence p_1, p_2, \dots , the corresponding summands s_1, s_2, \dots yield partial sums $s_1, s_1 + s_2, \dots$ formally converging to the same power series, namely the one with coefficient of x^n equal to $\sum \ell(f(w), v) y^{r(w)}$, summed over the mentioned finitely many pairs. Moreover, it follows that for any partition of p_1, p_2, \dots into subsequences $p_{1,j}, p_{2,j}, \dots$, $j = 1, 2, \dots$, we have the double counting identity $\sum_i s_i = \sum_i \sum_j s_{i,j} = \sum_j \sum_i s_{i,j}$. The identity in question is such double counting identity. On the one hand, grouping together the pairs (w, v) sharing the same weak SAW w we get the left hand side because for each w with $f(w) = uu' \in E(X) \setminus \partial X$ we have two contributions that cancel out ($\ell(uu', u) + \ell(uu', u') = 0$) and for each w with $f(w) = e \in \partial X$ we have a single contribution in which $r(w)$ depends only on e (by 2 of Proposition 4).

On the other hand, grouping together the pairs (w, v) sharing the same vertex v gives the right hand side of the identity. We fix $v \in X$ and split the group of the corresponding weak SAWs as

$$X(a, N(v)) = T_1 \cup T_2$$

where T_1 consists of the w containing one or two edges of $N(v)$ and T_2 of the w containing all three of them. Let $N(v) = \{e_1, e_2, e_3\}$ with a fixed counter-clockwise indexing.

Suppose that $w \in T_1$. If $E(w) \cap N(v)$ is $f(w)$ and the preceding edge, by deleting $f(w)$ we get another element of T_1 . If $E(w) \cap N(v)$ is just $f(w) = e_i$, we can always extend w by e_{i+1} and e_{i+2} of $N(v)$ and get an element of T_1 (now it is important to have weak SAWs and not just SAWs). Thus T_1 splits into triples w, w', w'' such that $E(w) \cap N(v) = \{e_i\} = \{f(w)\}$, so w is a SAW and v is its final vertex, and w' (resp. w'') is an extension of w by e_{i+1} (resp. by e_{i+2}). If $s = \ell(f(w), v) y^{r(w)} x^{|w|}$ is the summand corresponding to w , then due to additivity of winding and length and the multiplicative property of ℓ the summand of w' (resp. w'') is $s' = \zeta^2 y^{-1} x s$ (resp. $s'' = \zeta^4 y x s$). Taking s out and summing over all w and $v \in X$, we obtain the first term on the right side.

Suppose that $w \in T_2$. Let e_i be the edge of $N(v)$ appearing in w first. Then w splits in $w = w_1 w_2$ where w_1 is the initial part with final edge e_i and w_2 is a closed SAW with initial edge e_{i+1} and final edge e_{i+2} or vice versa. Reverting the direction of w_2 we get another element of T_2 . This operation is an involution that splits T_2 into pairs w, w' where w (resp. w') is obtained from a SAW w_1 with final vertex v and a cycle $C \in C(w_1)$ by setting v to be the initial/final vertex of C and choosing one of the two directions, which turns C

into a closed SAW w_2 (resp. w'_2). We may assume that the hook of w (resp. w') with the central vertex v is a left one (resp. a right one). By 1 of Proposition 4, $r(w) = r(w_1) + r(w_2) + 1 = r(w_1) + (r^*(w_2) - (-1)) + 1 = r(w_1) - 4$ and, similarly, $r(w') = r(w_1) + 4$ because w_2 is clockwise oriented (note that a and hence e_i lie in the exterior of w_2). We see that if t is the summand of w_1 then the summand of w (resp. w') is $s = \zeta^2 y^{-4} x^{|C|} t$ (resp. $s' = \zeta^4 y^4 x^{|C|} t$). Taking $x^{|C|} t$ out and summing over all w_1, C and $v \in X$, we obtain the second term on the right side. \square

Before we apply the identity to concrete domains, we point out two simplifications of its left side and discuss them in some generality. The first uses that $r(e)$ often depends only on the direction $\ell(\vec{e})$. We introduce some terminology. A biinfinite path p in H is a *quasiline* if left and right hooks alternate on it; such p separates H in two components. A SAW w is a *quasisegment* if $|w| \geq 2$ and is contained in a quasiline. A *quasihalfline* is defined in an obvious way. A set of vertices is *straight* if it forms a quasisegment, a quasihalfline or a quasiline. Any straight set of vertices has a unique extension to a quasiline. Suppose that $X \subset V$ is a domain and $a \in \partial X$. We say that a partition of border edges

$$\partial X \setminus \{a\} = A_1 \cup A_2 \cup \dots \cup A_k$$

is a *straight partition* if there exist vertex sets $U_i \subset X$, $i = 1, 2, \dots, k$, such that, for each i , U_i is straight and the edges in A_i are incident with U_i , lie all on one side of the quasiline extending U_i (and not on it) and form an interval (if $e, f \in A_i$ with endvertices $u, v \in U_i$ and $g \in E(X)$ is incident to U_i , lies on the same side of U_i as e and f and its vertex in U_i is between u and v , then $g \in A_i$).

Proposition 5 *Let $X \subset V$ be a simple domain, $a \in \partial X$ and $\partial X \setminus \{a\} = A_1 \cup A_2 \cup \dots \cup A_k$ be a straight partition. Then $e, f \in A_i$ implies that $\ell(\vec{e}) = \ell(\vec{f})$ and $r(e) = r(f)$, that is, $r(w) = r(w')$ whenever $w \in X(a, e)$ and $w' \in X(a, f)$. Thus the left side of the identity can be written as*

$$\ell(\vec{a})x + \sum_{i=1}^k \ell(A_i) y^{r(A_i)} F_{a, A_i}(x)$$

where $\ell(A_i) = \ell(\vec{e})$ and $r(A_i) = r(e)$ for any representing edge $e \in A_i$.

The proof is in Section 4. Further simplification uses the mirror symmetry along the x -axis. It is an automorphism of H and we denote it by M . Suppose that X is a simple domain such that $M(X) = X$, $a = (0, 0)(1, 0) \in \partial X$ and $\partial X \setminus \{a\} = A_1 \cup A_2 \cup \dots \cup A_k$ is a straight partition. Such partition can be usually defined so that all or almost all parts come in symmetric pairs. In any case, if A and $B = M(A)$, $A \neq B$, are two parts of such symmetric pair, we can pair their summands on the left side of the identity and get

$$\ell(A) y^{r(A)} F_{a, A}(x) + \ell(B) y^{r(B)} F_{a, B}(x) = \frac{\ell(A) y^{r(A)} + (\ell(A) y^{r(A)})^{-1}}{2} F_{a, A \cup B}(x) .$$

This is because M changes signs of winding numbers of SAWs and conjugates directions of border edges (in orientation away from X) and gives a length-preserving bijection between $X(a, e)$ and $M(X)(M(a), M(e)) = X(a, M(e))$ for any $e \in \partial X$.

Let us proceed with the lower bound on μ . If we specialize y to a complex number satisfying $\zeta^2 y^{-4} + \zeta^{-2} y^4 = 0$, the second term on the right side of the identity in Proposition 2 disappears. This equation has eight solutions

$$y_j = \zeta^{1/8} = \exp(2\pi i(6j+1)/48), \quad 0 \leq j \leq 7.$$

For exactly the same reason we can make almost vanish the left side of the identity by selecting appropriate X . We set

$$X = \mathbb{Z}^2 \setminus \{(p, q) \mid p \leq 0, p-1 \leq q \leq 1-p\} \quad \text{and} \quad a = (0, 0)(1, 0) \in \partial X.$$

We call X the *slit plane domain*. It is a simple domain, symmetric along the x -axis and its border edges, which are all except a vertical, have straight partition $\partial X \setminus \{a\} = A_1 \cup A_2$ where A_1 are the ones above the x -axis and $A_2 = M(A_1)$ are the symmetric ones. The corresponding set $U_1 \subset X$ is the quasihalfline with vertices $(0, 2), (0, 3), (-1, 3), (-1, 4), (-2, 4), (-2, 5), \dots$, and $U_2 = M(U_1)$ is the symmetric quasihalfline. It is easy to see that $\ell(A_1) = \zeta^4$ and $r(A_1) = 4$. Also, $\ell(\bar{a}) = -1$. By Proposition 5, the symmetry principle and the equation defining y_j , for each $y = y_j$ the left side of the identity for the slit plane domain equals

$$\sum_{e \in \partial X} \ell(\bar{e}) y_j^{r(e)} F_{a,e}(x) = \ell(\bar{a})x + \frac{(\zeta y_j)^4 + (\zeta y_j)^{-4}}{2} F_{a, A_1 \cup A_2}(x) = -x.$$

Corollary 6 *Let $X \subset V$ be the slit plane domain and $a = (0, 0)(1, 0) \in \partial X$. For each $l \in \{0, 1, \dots, 5\}$ and $k \in \{0, 1, \dots, 47\}$ we consider the generating function*

$$G_{l,k}(x) = \sum_{\ell(w)=\zeta^l, r(w) \equiv k \pmod{48}} x^{|w|}$$

where we sum over the SAWs w starting in a and with all vertices but the first in X . Then for each $j \in \{0, 1, \dots, 7\}$ we have in $\mathbb{C}[[x]]$ the formal identity

$$\frac{x}{1 + 2\operatorname{Re}(\zeta^4 y_j)x} = - \sum_{l=0}^5 \sum_{k=0}^{47} \zeta^l y_j^k G_{l,k}(x).$$

For $j = 7$ it follows that $G_{l,k}(x_c) = +\infty$ for some l, k and $x_c = \frac{1}{2\cos(\pi/8)}$.

Proof. We SAW that for $y = y_j$ and the slit plane domain X the identity of Proposition 2 turns in

$$-x = (1 + \zeta^{-4} x y_j^{-1} + \zeta^4 x y_j) \sum_{w \in S} \ell(w) y_j^{r(w)} x^{|w|}.$$

Rearranging the equation and using that each y_j is a (primitive) 48-th root of unity we get the stated formal identity. For $j = 7$ the real part of $\zeta^4 y_j = \exp(2\pi i(2/3 + 43/48)) = \exp(9\pi i/8)$ equals $-\cos(\pi/8)$. For $j = 7$ and $x = x_c = (2\cos(\pi/8))^{-1}$ the geometric series on the left side diverges, which implies that some of the 288 power series $G_{l,k}(x)$ on the right side diverges as well. \square

Thus $G_{l,k}(x) = \sum_{n \geq 1} a_n x^n$ with this special k and l satisfies $\limsup a_n^{1/n} \geq 2\cos(\pi/8)$ and so, since $s_n \geq a_n$, $\mu \geq 2\cos(\pi/8)$.

3 The upper bound on μ

We show that s_n , the number of n -step SAWs in H starting at the origin, has for $n = 1, 2, \dots$ and every $\delta > 0$ bound

$$s_n \ll_\delta (x_c^{-1} + \delta)^n = (2\cos(\pi/8) + \delta)^n.$$

This implies that $\mu \leq 2\cos(\pi/8)$.

We apply Proposition 2 to finite trapezoidal domains that we now define. For integers $r, s \geq 2$ we set

$$X_{r,s} = \{(p, q) \in \mathbb{Z}^2 \mid 1 \leq p \leq r, |q| \leq 2s + p - 2\} \text{ and } a = (0, 0)(1, 0) \in \partial X_{r,s}.$$

$X = X_{r,s}$ is a simple domain, symmetric along the x -axis and its border edges have straight partition

$$\partial X \setminus \{a\} = A^+ \cup B^+ \cup C \cup B^- \cup A^-$$

where A^+ are the horizontal edges above a , B^+ are the vertical edges above the x -axis, C are the horizontal edges to the right of X and $B^- = M(B^+)$ and $A^- = M(A^+)$ are the symmetric parts. The corresponding five quasisegments in X are, in the respective order, $U_1 = \{(1, q) \mid 1 \leq q \leq 2s - 1\}$, $U_2 = \{(p, 2s + p - 2), (p, 2s + p - 3) \mid 1 \leq p \leq r\}$, $U_3 = \{(r, q) \mid |q| \leq 2s + r - 2\}$ and the symmetric ones $U_4 = M(U_2)$ and $U_5 = M(U_1)$. Clearly, $\ell(\bar{a}) = -1$, $\ell(A^+)y^{r(A^+)} = -y^3$, $\ell(B^+)y^{r(B^+)} = \zeta^2 y^2$ and $\ell(C)y^{r(C)} = 1$. Denoting $A = A^+ \cup A^-$ and $B = B^+ \cup B^-$, we get the next result, Lemma 2 of [1].

Corollary 7 *Let $x_c = \frac{1}{2\cos(\pi/8)}$. For all integers $r, s \geq 2$ we have for the domain $X = X_{r,s}$ with border edges partitioned into $\{a\} \cup A \cup B \cup C$ the identity*

$$\cos(3\pi/8)F_{a,A}(x_c) + \cos(\pi/4)F_{a,B}(x_c) + F_{a,C}(x_c) = x_c.$$

Proof. In Proposition 2 we set $X = X_{r,s}$, $y = y_7$ and $x = x_c$, which is admissible since all sums are finite. Then both factors on the right side of the identity are zero and the right side vanishes completely. By Proposition 5 and by symmetry we obtain equation

$$-x_c - \frac{y_7^3 + y_7^{-3}}{2} F_{a,A}(x_c) + \frac{\zeta^2 y_7^2 + \zeta^{-2} y_7^{-2}}{2} F_{a,B}(x_c) + F_{a,C}(x_c) = 0.$$

The coefficient of $F_{a,A}(x_c)$ equals $\cos(3 \cdot 2\pi(43/48)) = -\cos(3\pi/8)$ and that of $F_{a,B}(x_c)$ equals $\cos(2 \cdot 2\pi(1/6 + 43/48)) = \cos(\pi/4)$. Rearrangement gives the stated formula. \square

It only is important that $\cos(3\pi/8)$, $\cos(\pi/4)$ and x_c are positive and the involved power series have nonnegative coefficients. This identity will provide the crucial estimate.

We define the set of border edges $D \subset \partial X_{r,s}$ as $D = B \cup C \cup \{e, f\}$ where e is the top edge of A^+ and f the bottom edge of A^- . Note that D are exactly the border edges incident with the set $Y \subset X_{r,s}$, defined as $Y = U_2 \cup U_3 \cup U_4$. We call $U_1 \cup \{(1,0)\} \cup U_5$ the *left side*, U_2 the *top side*, U_3 the *right side* and U_4 the *bottom side* of $X_{r,s}$. Note that each SAW starting at a needs at least s steps to reach the top or bottom side of $X_{r,s}$ and at least r steps to reach the right side. We have this stratification property of trapezoids: If $X = X_{r,s}$, $X' = X_{r-1,s}$ and $X'' = X_{r,s-1}$, then $X', X'' \subset X$, $X \setminus X'$ is the right side of X and $X \setminus X''$ is the union of the top and bottom side of X .

Let $a = (0,0)(1,0)$ be the initial edge as before. We bound s_n from above by considering four classes of SAWs in H with decreasing degree of generality and by counting their SAWs by length n . All SAWs starting at $(0,0)$ are counted by s_n . The *SAWs starting at a* start at $(0,0)$, have initial edge a and are counted by b_n . *Halfplane SAWs*, counted by c_n , are those of them that after the initial edge a stay in the right halfplane $x \geq 1$. *Trapezoidal SAWs*, counted by d_n , is the further subset

$$\bigcup_{r,s=2}^{\infty} X_{r,s}(a, D)$$

where the trapezoids $X_{r,s}$ and sets of border edges $D = D_{r,s}$ were defined above. For example, $d_1 = d_2 = d_3 = 0$ and $d_4 = 2$. We estimate the numbers s_n, b_n, c_n and d_n .

For this we employ and first recall the symmetry properties of H . For every two edges $e, f \in E$ and their endvertices $u \in e, v \in f$ there exist exactly two automorphisms of H sending u to v and e to f , one preserving the counter-clockwise order of edges in $N(t)$ for any vertex t and the other reversing it. The former automorphisms are of the *first kind* and the latter of the *second kind*. The inverse to an automorphism is clearly of the same kind. For example, the mirror symmetry along the x -axis M is of the second kind and is an involution. In the original hexagonal lattice graph H' automorphisms of the first kind are induced by shifting the plane to identify u with v and then rotating it around v to identify the shifted e with f ; automorphisms of the second kind need additional mirror symmetry along the line going through f . If p is a quasiline in H and e is an edge incident with p but not on p , we denote by $H(p, e)$ the set of SAWs w in H starting at e and such that p separates e and the remaining edges of w . Clearly, both automorphisms sending $v = e \cap p$ to the origin and e to a provide a length-preserving bijection between $H(p, e)$ and the set of halfplane SAWs defined above. As we know, the border edges in $A \cup \{a\}$, B^+ , C and B^- are separated from $X_{r,s}$ by the quasiline extending, respectively, the left side,

the top side, the right side and the bottom side of $X_{r,s}$. Thus $H(p, e)$, for p extending a side of $X_{r,s}$ and e in the corresponding set of border edges, is in length-preserving bijection with halfplane SAWs. Also, every SAW starting in such e and with all vertices but the first in $X_{r,s}$ lies in the set $H(p, e)$.

Let us relate the numbers s_n, b_n, c_n and d_n . We have the simple bounds

$$s_n = 3b_n \quad \text{and} \quad b_n \leq c_{n+1} + \sum_{i=1}^{n+3} c_i c_{n+4-i}, \quad n = 1, 2, \dots$$

The equality follows from the fact that every edge incident with the origin is sent by an automorphism fixing the origin to a . As for the inequality, for a SAW w , $|w| = n \geq 1$, starting at a consider the rightmost vertical quasiline p containing its vertex and take the topmost of such vertices v . We distinguish two cases depending on whether v is incident to an edge e to the right of p or not. If v is incident to such edge e then v is the last vertex of w for else the hook of w with the central vertex v would be in p and v would not be the topmost vertex of w on p . We extend w by the one-step SAW $w_0 = e$. In the other case we step from v first up and then to the right of p (this does not revisit w) and let w_0 be the corresponding two-step SAW. In the first case we set $w_1 = F((ww_0)^r)$ and in the second $w_2 = F((w'w_0)^r)$ and $w_3 = F(w_0^r w'')$ where w' is the initial part of w ending in v , w'' (which may be empty) is the final part of w starting in v and F is the unique automorphism of H , say of the first kind, that sends the final vertex of w_0 to the origin and $f(w_0)$ to a . All three SAWs w_1, w_2 and w_3 are halfplane SAWs because $(ww_0)^r, (w'w_0)^r$ and $w_0^r w''$ are in $H(p, f(w_0))$. Thus $w \mapsto w_1$ or $w \mapsto (w_2, w_3)$ is an injective mapping sending w to a halfplane SAW or to a pair of them and $|w_1| = n+1$ or $|w_2| + |w_3| = n+2|w_0| = n+4$. It is injective because if w_1 or (w_2, w_3) is in the image of the mapping, w is reconstructed by applying the unique automorphism F' of the first kind sending the origin to the last vertex of w_1 or w_2 and a to $f(w_1)$ or $f(w_2)$ and by omitting the last edge of $F'(w_1)$ or the two edges of $F'(w_2) \cap F'(w_3)$. This proves the inequality.

The breakthrough bound

$$d_n < 4nx_c^{-n}, \quad n = 1, 2, \dots,$$

follows from Corollary 7. Consider a trapezoidal SAW $w \in X_{r,s}(a, D)$ with length n . If $f(w) \in A \cup B$ then, by definition of $X_{r,s}$, $w \in X_{n,s}(a, D)$ and if $f(w) \in C$ then $w \in X_{r,n}(a, D)$. Thus all trapezoidal SAWs with length n are contained in the union of $2n - 2$ sets

$$\bigcup_{r=2}^n X_{r,n}(a, D) \cup \bigcup_{s=2}^n X_{n,s}(a, D).$$

For fixed r and s , the number of SAWs $w \in X_{r,s}(a, D)$ with length n is less than x_c^{-n} because the identity in Corollary 7 implies that $F_{a,D}(x_c) \leq F_{a,A}(x_c) + F_{a,B}(x_c) + F_{a,C}(x_c) \leq x_c / \cos(3\pi/8) < 2$ and the coefficient of x^n in $F_{a,D}(x)$ cannot be $\geq 2x_c^{-n}$. Summing over the $2n - 2$ sets of SAWs gives the bound.

To close the chain we bound c_n in terms of d_n , halfplane SAWs in terms of trapezoidal SAWs. Arguing as before we derive the recurrent inequality

$$c_n \leq \sum_{i=\sqrt{n/6}}^{n+1} d_i c_{n+2-i} + \sum_{i=\sqrt{n/6}}^{n+3} d_i c_{n+4-i}, \quad n = 3, 4, \dots$$

Let w be a halfplane SAW with $|w| = n \geq 3$. Consider the trapezoid $X_{r,s}$ containing w (except the initial vertex) that has the minimum size $|X_{r,s}|$; we have $|X_{r,s}| \leq r(4s + 2r - 3) < 6 \max(r, s)^2$. Due to the minimality of $X_{r,s}$, $|w| \geq 3$ and the stratification property of trapezoids, w visits the right side of $X_{r,s}$ or the union of its top and bottom side. If only the former visit occurs then $s = 2$ and if only the latter visit occurs then $r = 2$. In any case, if v_j is the last vertex of w in $Y = Y_{r,s}$ then $j \geq \max(r, s)$ because w needs so many steps to reach the right side and the top or bottom side. Also, $\max(r, s) \geq \sqrt{n/6}$ for else $|X_{r,s}| < n$ and w would not fit in $X_{r,s}$. Thus $j \geq \sqrt{n/6}$. As above we distinguish two cases depending on whether v_j is incident with a border edge in D or not. If v_j is incident with such edge $e \in D$, we set $w_0 = e$ (oriented away from $X_{r,s}$), $w_1 = w'w_0$ and $w_2 = F(w_0^r w'')$ where w' is the initial part of w ending in v_j , w'' (which may be empty) is the final part of w starting in v_j and F is the unique automorphism of H , say of the first kind, that sends the final vertex of w_0 to the origin and $f(w_0) = e$ to a . (Unlike the bound for b_n and c_n , here if $e \in A$ then v_j need not be the last vertex of w .) If v_j is not incident with a border edge in D , it has (at least) two neighbors in $Y_{r,s}$ and each of them is incident with an edge in D . It follows that one of these neighbors is not in w , for else both would be earlier vertices of w and both hooks in w with these central vertices would use edges incident with v_j , both of which would be in the natural orientation oriented towards v_j , which is impossible. Hence there exists a two-step SAW w_0 starting in v_j , not revisiting w and with final edge in D . We define $w_3 = w'w_0$ and $w_4 = F(w_0^r w'')$ where w', w'' and F are as in the first case. Clearly, $|w_1| + |w_2| = n + 2$, $|w_3| + |w_4| = n + 4$, w_1 and w_3 are trapezoidal SAWs, $|w_1|, |w_3| \geq j \geq \sqrt{n/6}$, and w_2 and w_4 are halfplane SAWs because $w_0^r w''$ is in $H(p, f(w_0))$ where p is the quasiline containing a side of $X_{r,s}$ and separating $X_{r,s}$ from the last vertex of w_0 . The mapping $w \mapsto (w_1, w_2)$ or $w \mapsto (w_3, w_4)$ is injective because one inverts it by applying the unique automorphism of the first kind F' sending the origin to the last vertex of w_2 or w_4 and a to $f(w_2)$ or $f(w_4)$ and by deleting the one or two edges of $F'(w_2) \cap w_1$ or $F'(w_4) \cap w_3$. This proves the inequality.

Combining the bounds we prove the upper bound on s_n stated at the beginning of this section. We prove, using the last two bounds on c_n and d_n , that

$$c_n \ll_\delta (x_c^{-1} + \delta)^n, \quad n = 1, 2, \dots, \quad \delta > 0,$$

(we use \ll as synonymous to $O(\cdot)$, not to $o(\cdot)$). The simple bounds on s_n and

b_n then imply

$$\begin{aligned} s_n &= 3b_n \ll_{\delta} 3((x_c^{-1} + \delta)^{n+1} + (n+3)(x_c^{-1} + \delta)^{n+4}) \\ &\ll_{\delta} (x_c^{-1} + \delta)^n = (2 \cos(\pi/8) + \delta)^n, \text{ for any } \delta > 0. \end{aligned}$$

To upperbound c_n , we fix a $\delta \in (0, 1)$ and use induction on n . We take an $n_0 > 100$ such that $n > n_0$ implies that $(x_c^{-1} + 1)^4(8n^2 + 16n) < (1 + \delta x_c)\sqrt{n/6}$ and then a constant $\kappa = \kappa(\delta) > 1$ such that $c_n < \kappa$ for $n = 1, 2, \dots, n_0$. We claim that for every $n = 1, 2, \dots$,

$$c_n < \kappa(x_c^{-1} + \delta)^n.$$

For $n \leq n_0$ this is true. For $n > n_0$ we use the recurrent bound on c_n , the upper bound on d_n , the inductive assumption for c_{n+2-i} and c_{n+4-i} and the property of n_0 and get that

$$\begin{aligned} c_n &\leq \sum_{i=\sqrt{n/6}}^{n+1} d_i c_{n+2-i} + \sum_{i=\sqrt{n/6}}^{n+3} d_i c_{n+4-i} \\ &< \kappa \sum_{i=\sqrt{n/6}}^{n+1} 4ix_c^{-i}(x_c^{-1} + \delta)^{n+2-i} + \kappa \sum_{i=\sqrt{n/6}}^{n+3} 4ix_c^{-i}(x_c^{-1} + \delta)^{n+4-i} \\ &< \kappa(x_c^{-1} + 1)^4(8n^2 + 16n)(1 + \delta x_c)^{-\sqrt{n/6}}(x_c^{-1} + \delta)^n \\ &< \kappa(x_c^{-1} + \delta)^n \end{aligned}$$

as well. This concludes the proof of the upper bound on s_n and on μ . Theorem 1 is proved.

4 Proofs of Propositions 4 and 5 on winding number

Proof of part 1 of Proposition 4. Let $w = (v_1, e_1, \dots, e_n, v_{n+1} = v_1)$ be a closed SAW in the hexagonal graph H . The *width* of w is $s = a_2 - a_1$ where a_2 (resp. a_1) is the maximum (resp. minimum) x -coordinate of a vertex in w . A *bulge* in w is any SAW formed by an interval of edges e_i, e_{i+1}, \dots, e_j of w with $1 \leq i < j \leq 2n-1, 2 \leq j-i \leq n-1$ (indices $> n$ are understood as reduced modulo n) and such that $e_{i+1}, e_{i+2}, \dots, e_{j-1}$ are all vertical, lying on the line $x = a$, and e_i, e_j are both horizontal, lying to the right of $x = a$. Note that $w \cap (x = a_1)$ consists of several (and at least one) disjoint intervals of edges, each of which creates a bulge. Hence w contains at least one bulge.

We proceed by induction on $n = |w| \geq 6$. If $s = 1$ then w is the perimeter of a rectangle R with unit width and height $2h$. Traversing the perimeter of R counter-clockwise we see that the vertices on each vertical side determine $h+2$ left and $h-1$ right hooks. Thus the weight of all hooks on the perimeter is

$2(h+2-(h-1)) = 6$. If the perimeter of R is traversed clockwise, its hooks have weight -6 . So in this case $r^*(w) = \pm 6$, with sign determined by the orientation of w .

Let $s > 1$. We consider a bulge $B = \{e_i, e_{i+1}, \dots, e_j\}$ in w with the maximum value of a . None of the horizontal edges of H between e_i and e_j is in w . Let I be the interval of vertical edges of H on the line $x = a + 1$ between e_i and e_j . The edges in I and not in w form a single nonempty interval $J \subset I$. Indeed, the whole I cannot be in w (then we would have $s = 1$) and there cannot be two such (neighboring) intervals J , because their separating interval of edges in $I \cap E(w)$ would create a bulge contradicting the maximality of a . We obtain a shorter closed SAW w' by deleting from w the edges in $B \cup (I \cap E(w))$ and adding the edges in J (oriented compatibly with the rest of w). Since w' is 2-regular, connected and self-avoiding, it is indeed a closed SAW. Clearly, $|w'| \leq |w| - 2$. To conclude the argument by induction on length we need to verify that w and w' have the same orientation and $r^*(w) = r^*(w')$.

The exchange of edges does not alter orientation because the rightmost vertical edges of w on $x = a_2$ lie to the right of $x = a + 1$ and are not affected by the exchange, keep their orientation, and the exterior of w and w' lies to the right of them. We calculate the change in r^* . Let w_B be the rectangular closed SAW with edges $B \cup I$ (with orientation inherited from w) and U be its vertex set. We have partition $U = U_1 \cup \{b, t\} \cup U_2$ where U_1 are the inner vertices of J , b and t are the endvertices of J and U_2 are the remaining vertices. For any $v \in U$ we let $h(v)$ be the hook in w_B with the central vertex v . Subscripts w and w' , for example in $h(b)_w$, define hooks with the given central vertex and contained in the subscripted SAW. We have

$$\begin{aligned}
r^*(w) &= r^*(w') - r(h(b)_{w'}) - r(h(t)_{w'}) - \sum_{v \in U_1} r(h(v)_{w'}) + \\
&\quad + r(h(b)_w) + r(h(t)_w) + \sum_{v \in U_2} r(h(v)_w) \\
&= r^*(w') - r(h(b)_{w'}) - r(h(t)_{w'}) + r(h(b)_w) + r(h(t)_w) + \\
&\quad + \sum_{v \in U_1 \cup U_2} r(h(v)) \\
&= r^*(w') + \sum_{v \in \{b, t\}} (r(h(v)_w) - r(h(v)_{w'}) - r(h(v))) + \sum_{v \in U} r(h(v)) \\
&= r^*(w') + \sum_{v \in \{b, t\}} (r(h(v)_w) - r(h(v)_{w'}) - r(h(v))) \pm 6,
\end{aligned}$$

with the sign $+$ iff w_B is counter-clockwise oriented, because the last but one sum is $r^*(w_B)$ and was handled in the case $s = 1$. In the last sum we have $r(h(b)) = r(h(t))$ and the same for terms with subscripts because in the pair of hooks in question one is obtained from the other by mirroring it along the bisector of the segment J and then reverting its direction. The definition of w' from w via exchange of edges shows that $r(h(v)_{w'}) = -r(h(v)_w)$ —the edge of $N(v)$ common to both hooks has in them the same orientation. In the hooks

$h(v)_{w'}$ and $h(v)$ the common edge (which is an extremal edge of J) is oriented differently, one hook enters by it and the other leaves, and so $r(h(v)_{w'}) = r(h(v))$. Finally, the sign of $r(h(t))$, say, equals to that of the last term since $h(t)$ turns left iff w_B is counter-clockwise oriented. Altogether this shows that the last sum equals ∓ 6 and cancels with the last term. Thus $r^*(w) = r^*(w')$.

Proof of part 2 of Proposition 4. Let $X \subset V$ be a simple domain, $a, e \in \partial X$ be two boundary edges and $w_1, w_2 \in X(a, e)$ be two SAWs (possibly closed). We distinguish two cases according to the number of components of $H \setminus X$ the edges a and e lie in, by which we mean more precisely their endvertices in $V \setminus X$. The first case is when a and e lie in the same connected component K of $H \setminus X$. Then there is a SAW w going from e to a in K . The unions $w \cup w_1$ and $w \cup w_2$ are closed SAWs which are moreover oriented in the same sense. Suppose for contrary that they have different orientations, one clockwise and the other counter-clockwise. Then, by Proposition 3, each vertex in K but not on w is in the interior of $w \cup w_1$ or $w \cup w_2$. It follows that for any vertex v in K no matter how we walk from v to infinity the walk has to intersect $w \cup w_1 \cup w_2$. But this means that K is bounded and hence finite, which was excluded. Thus, by part 1, $r^*(w \cup w_1) = r^*(w \cup w_2) = \pm 6$ and from $r^*(w \cup w_1) = r(w) + r(w_1)$ and $r^*(w \cup w_2) = r(w) + r(w_2)$ we get that $r(w_1) = r(w_2)$.

The second case is when a is in K and e is in L where K and L are two distinct connected components of $H \setminus X$. We will be done as in the previous case if we find a SAW w in H going from e to a such that it does not intersect, except the overlaps in e and a , the SAWs w_1 and w_2 and the resulting closed SAWs $w \cup w_1$ and $w \cup w_2$ have the same orientation. It is not hard to obtain such w using that K and L are infinite and thus unbounded. Either there exist vertices in K with x -coordinate going to $+\infty$ or to $-\infty$ or the same holds with y -coordinate, and similarly for L . Let us consider the case that both K and L contain vertices with y -coordinate going to $-\infty$; the other 15 cases are similar. We fix $c \in \mathbb{Z}$ so small that both w_1 and w_2 lie above the line $y = c$. Then we walk from a in K down by a SAW w_3 until we hit the line $y = c$ and do the same in L with a SAW w_4 starting in e . Except the final vertices both w_3 and w_4 lie above $y = c$ and do not intersect as they lie in distinct components of $H \setminus X$. We walk from the final vertex of w_4 to that of w_3 by a SAW w_5 that lies on or below $y = c$ and hence besides these final points w_5 is disjoint to $w_1 \cup w_2 \cup w_3 \cup w_4$. Thus $w = w_3 \cup w_4 \cup w_5$ is the required SAW. If we close it by w_1 or w_2 , the orientation is the same because below the lowest horizontal edges of w_5 always lies the exterior of $w \cup w_1$ and $w \cup w_2$.

Proof of Proposition 5. Let X be a simple domain, $a \in \partial X$, $e, f \in A$ be two distinct border edges in one part A of a straight partition of $\partial X \setminus \{a\}$ and U be the corresponding straight set of vertices of X . We may assume that U is a quasisegment joining the vertices of e and f in X ; e and f lie on the same side of U . We make U a SAW by orienting it from e to f . Then left and right hooks alternate on eUf except at the ends where the first two and last two hooks have, all three or four, the same orientation. It follows from this that in the natural orientation $\ell(\bar{e}) = \zeta^3 \ell(\bar{f}) = -\ell(\bar{f})$. Hence e and f have the same direction when oriented away from X .

We show that $r(e) = r(f)$. Let w_0 be the shortest SAW joining a to U in X . Thus w_0 is disjoint to U except that the final vertex $v \in U$. We set $w_1 = w_0 U' e$ and $w_2 = w_0 U'' f$ where U' is the part of U between v and the endvertex of e and similarly for U'' . So $w_1 \in X(a, e)$ and $w_2 \in X(a, f)$. Let h_1 (resp. h_2) be the hook in w_1 (resp. w_2) with the central vertex v . If v is an extremal vertex of U , say an endvertex of e , we have $w_1 = w_0 e$, $w_2 = w_0 U f$ and $r(w_2) = r(w_1) - r(h_1) + (-r(h_1)) + r(U) + r(h_1) = r(w_1) + r(U) - r(h_1) = r(w_1)$ because U has an odd number of hooks which alternate on it between left and right and h_1 has the same orientation as the two extremal hooks of U . If v is an inner vertex of U , $U = (U')^r U''$ and, by the definition of straight partition, $f(w_0)$ lies on the other side of U than e and f . Thus, denoting by h the hook in U with the central vertex v , we see that again $r(w_2) = r(w_1) - r(h_1) - r(U' e) + r(h_2) + r(U'' f) = r(w_1) + r(e(U')^r) + r(U'' f) + r(h) - r(h) - r(h_1) + r(h_2) = r(w_1) + r(e U f) - r(h) - r(h_1) + r(h_2) = r(w_1) + r(e U f) - 3r(h) = r(w_1)$. So $r(w_1) = r(w_2)$ and, by 2 of Proposition 4, $r(e) = r(f)$. \square

References

- [1] H. Duminil-Copin and S. Smirnov, The connective constant of the honeycomb lattice equals $\sqrt{2 + \sqrt{2}}$, arXiv:1007.0575v1, 4 July 2010, 9 pages.
- [2] A. J. Guttmann (Ed.), *Polygons, Polyominoes and Polycubes*, Lecture Notes in Physics, Springer, 2009.
- [3] T. C. Hales, The Jordan curve theorem, formally and informally, *Amer. Math. Monthly* **114** (2007), 882–894.
- [4] T. C. Hales, Jordan’s proof of the Jordan curve theorem, *Studies in Logic, Grammar and Rhetoric* **10 (23)** (2007), 45–60.
- [5] B. Nienhuis, Exact critical points and critical exponents of $O(n)$ models in two dimensions, *Phys. Rev. Lett.* **49** (1982), 1062–1065.